Fractional Calculus in APL

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Introduction

This topic shares a philosophic relationship with Fractals, which extend the familiar integer-only dimensions such as Length (dimension 1), Area (dimension 2), and Volume (dimension 3) to fractional values. The impetus for Fractals is that they are better at describing certain real-world phenomena than the integer-only dimensions. For example, Mandelbrot titled his seminal paper **“How Long Is the Coast of Britain?”** because he wanted to make the point that length is an inappropriate measure for that real-world question. In fact, the actual fractal dimension of the coast of Britain is about 1.25, that is, about a quarter of the way between Length and Area.

Similarly, Fractional Calculus extends the familiar integer-only derivatives such as Velocity (1^{st} derivative), Acceleration (2^{nd} derivative), and Kick (3^{rd} derivative), and the corresponding integrals (as an inverse or anti-derivative) to fractional values. The impetus for Fractional Calculus is that it is better at describing certain real-world phenomena than the integer-only derivatives/integrals. For example, it makes sense to talk about a half derivative, as in `∂[0.5] R`, and in the sense that `∂[0.5] ∂[0.5] R ↔ ∂R`, that is, two half derivatives equal a full derivative.

Notation

The chosen symbol for derivatives (Partial Differential, ∂, Unicode 0x2202) is extended to accept an axis operator which specifies the power of the derivative, as in `f ∂[χ] R`. Similarly, the chosen symbol for integrals (Integral, ∫, Unicode 0x222B) is extended the same way, as in `L f ∫[χ] R`.

Implementation

There are many different definitions for Fractional Integrals[4]. The one I chose is the "classical" form of Fractional Calculus known as the "Riemann-Liouville fractional integral". The easiest way to implement this definition is via the identity, known as the "**Cauchy formula for repeated integration**"[2], which states:

$$f^{(-n)}(x)=\frac{1}{(n-1)!}\int_a^x (x-t)^{n-1} f(t)\,dt$$

where $f^{(-n)}$ is the notation for f **differentiated** $-n$ times, which really means f **integrated** n times.

In other words, when n is a positive integer, rather than execute repeated integration (`∫∫...∫`) n times, only one integration is needed. As it turns out, this same identity easily generalizes to fractional values of the power n. In fact, it also generalizes to Hypercomplex numbers, but alas, I don't have the necessary code to integrate or differentiate complex numbers, no less Hypercomplex ones.

Translated into APL, this identity looks like

```
L f ∫[χ] R ↔ (L f{(αα ω)×(ωω-ω)*χ-1}[χ] R∫R)÷!χ-1
```

where Chi (χ) is the axis operator as passed into anonymous functions, and if L (the lower limit of integration) is zero, it may be omitted.

As with Fractional Integrals, there are many different definitions[5] of Fractional Derivatives. To be consistent with Fractional Integrals, I selected the same Riemann-Liouville definition for Fractional Derivatives.

```
(int frc)←0 1⊤χ ⍝ Separate integer and fraction
⍝ Integrate up the fraction power
⍝  and then differentiate down the integer power
L f ∂[χ] R ←→ L f ∂[int+1] ∫[1-frc] R
```

Note that the higher power derivative part (`∂[int+1]`) is non-iterative and fast because I found a clever algorithm[6] to do Numerical Derivatives. It uses the power of the derivative to index a table to obtain a set of coefficients to run through their algorithm avoiding iteration through successive powers.

There is also special code used when one or both of the Integration limits is infinite – this case must be handled specially, because the Numerical Integration code I use, assumes that all limits are finite. For these three cases, I use the three identities found in Wikipedia[3] which through a clever change of variables, transforms the integration over an infinite range to a finite interval:

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-1}^{+1} f\left(\frac{t}{1-t^2}\right)\frac{1+t^2}{(1-t^2)^2}\,dt$$

$$\int_{a}^{\infty} f(x)\,dx = \int_{0}^{1} f\left(a+\frac{t}{1-t}\right)\frac{dt}{(1-t)^2}$$

$$\int_{-\infty}^{a} f(x)\,dx = \int_{0}^{1} f\left(a-\frac{1-t}{t}\right)\frac{dt}{t^2}$$

Applications

To quote one of many, many articles about Fractional Calculus:

"The fractional calculus deals with extensions of derivatives and integrals to noninteger orders. It represents a powerful tool in applied mathematics to study a myriad of problems from different fields of science and engineering, with many break-through results found in mathematical physics, finance, hydrology, biophysics, thermodynamics, control theory, statistical mechanics, astrophysics, cosmology and bioengineering [8] [9] [10] [11] ."[7]

However, simple applications on the order of "**How Long Is The Coast of Britain?**" are not easy to find. They are more about, say, "Application to the differentiation of fractal curves" which, not surprisingly, contains many references to Mandelbrot who knew about and used Fractional Calculus.

Please consult the references for more details.

To Do

I would like to extend both Integer and Fractional Calculus to Hypercomplex numbers, once I understand the theory behind it.

Acknowledgments

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References

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